Effects of Fringing Fields on the Capacitance of Circular Microstrip Disk

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Abstract—The effects of fringing fields on the capacitance of a circular microstrip disk are studied with the dual integral equation formalism for practical microstrip circuits when the substrate thickness is small. Approximations as well as exact numerical evaluations are made in the calculation of the capacitance. With a seminumerical approach, an approximate formula for the capacitance is obtained and shown to yield accurate results with the simple use of a calculator. Asymptotic lower bound and exact numerical computations are also carried out. The various techniques are illustrated and compared with numerical results.

I. INTRODUCTION

THE FRINGING field effects on the capacitance of a circular parallel-plate capacitor has been of historical interest and more recently has become a fashionable topic because of the applications to microstrip circuit and antenna elements. An approximate solution for the capacitance of a circular capacitor in free space was first obtained by Kirchhoff [1] in 1877 by making use of the technique of conformal mapping to account for the fringing fields. In 1932 Ignatowsky [2] obtained the capacitance in the limit of small plate separation, whose result was later shown by Polya and Szegö [3] with a variational technique to be an asymptotic lower bound. In 1963 Hutson [4] demonstrated rigorously the validity of Kirchhoff's formula. Leppington and Levine [5] in 1970 used an integral equation of the first kind for the distribution of the potential off the disks to obtain an approximation for the capacitance, reproducing the result of Kirchhoff and Hutson and providing details with regard to the next correction term. Of all the results that confirmed Kirchhoff's result, conformal mapping was used in computing the fringing field effects.

In microstrip applications, the capacitor plates are separated by a dielectric sheet instead of free space. Employing Galerkin's method and using a basis function which corresponds to a constant charge distribution, Itoh and Mittra [7] obtained numerically the capacitance for a circular microstrip disk. Borkar and Yang [9] formulated the problem in terms of dual integral equations and numerically computed the capacitance for small plate separation. For practical usage, Shen *et al.* [10] devised an approximate formula for the capacitance by drawing analogy with Kirchhoff's formula.

In this article we make use of the dual integral equations formulation to study the effects of fringing fields on the capacitance of a circular microstrip disk. We first derive an asymptotic lower bound for the capacitance reducible to that obtained by Polya and Szegö [3] in the limit of free space. We then evaluate with a numerical method the capacitance for small plate separation. For practical usage, we also obtain, with a seminumerical approach, an approximate formula for the capacitance of the microstrip disk. The various techniques are compared and illustrated with various numerical results.

II. FORMULATION AND ASYMPTOTIC RESULT

Consider a circular microstrip disk with radius a at the potential of V and separated from the ground plane by a dielectric substrate with relative permittivity ϵ_r and thickness d (Fig. 1). The mixed boundary value problem has been formulated in terms of dual integral equations [7], [9]

$$\int_0^\infty d\alpha\, \alpha \hat{\sigma}(\alpha) G(\alpha) J_0(\alpha\rho) = \frac{V\epsilon_r\epsilon_0}{d}, \qquad 0<\rho<a \quad (1a)$$

and

$$\int_0^\infty d\alpha\, \alpha \hat{\sigma}(\alpha) J_0(\alpha\rho) = 0, \qquad \rho>a \quad (1b)$$

where

$$G(\alpha) = \frac{\epsilon_r\left[1 - e^{-2\alpha d}\right]}{(1+\epsilon_r)\alpha d\left[1 - ((1-\epsilon_r)/(1+\epsilon_r))e^{-2\alpha d}\right]} \quad (2)$$

and

$$\sigma(\rho) = \int_0^\infty d\alpha\, \alpha \hat{\sigma}(\alpha) J_0(\alpha\rho) \quad (3)$$

is the charge distribution on the disk.

Applying a method similar to that of Galerkin [7], Tranter [11] assumed that

$$\hat{\sigma}(\alpha) = \alpha^{-\kappa} \sum_{m=0}^\infty a_m J_{2m+\kappa}(\alpha a). \quad (4)$$

Substituting (4) into (3), we obtain

$$\sigma(\rho) =$$

$$\begin{cases} \dfrac{\left\{\left[1-(\rho/a)^2\right]/2\right\}^{\kappa-1}}{a^{2-\kappa}} \displaystyle\sum_{m=0}^\infty a_m \dfrac{\Gamma(m+1)}{\Gamma(m+\kappa)} P_0^{(0,\kappa-1)}\left[1-2\left(\dfrac{\rho}{a}\right)^2\right], \\ \hspace{7cm} \rho<a \\[4pt] 0, \hspace{5cm} \rho>a \end{cases} \quad (5)$$
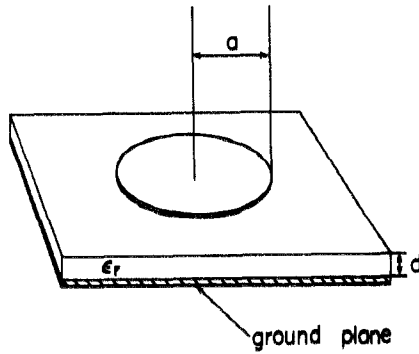
Fig. 1. Geometrical configuration of the problem.

where $P_m^{(\alpha,\beta)}(x)$ is a Jacobi's polynomial. By doing so, equation (1b) is automatically satisfied. Substitution of (4) into (1a) produces an infinite system of linear equations which can be solved by an iteration process. It is appropriate to choose $\kappa = 1$ so that the iterative solution converges quickly when $d \to 0$. When $\kappa = 1$, the Jacobi's polynomial becomes Legendre's polynomial, thus the above approach is similar to that used by Polya and Szegö [3].

The infinite system of linear equations can be written as

$$\sum_{m=0}^{\infty} a_m A_{mn} = E(n), \qquad n = 0, 1, 2, \cdots \qquad (6a)$$

where

$$A_{mn} = \int_0^{\infty} d\alpha \, G(\alpha) \alpha^{-1} J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) \qquad (6b)$$

and

$$E(n) = \begin{cases} 0, & n > 0 \\ V \epsilon_r \epsilon_0 a / 2d, & n = 0. \end{cases} \qquad (6c)$$

Furthermore, by integrating the charge density in (5), it can be shown [3], [9] that the capacitance of the disk is

$$C = \frac{\pi a}{V} a_0. \qquad (7)$$

To find a small d approximation for A_{mn}, we rewrite it as [11]

$$A_{mn} = \frac{\delta_{mn}}{(2+4n)} + B_{mn} \qquad (8a)$$

where δ_{mn} is the Kronecker delta function and

$$B_{mn} = \int_0^{\infty} d\alpha [G(\alpha) - 1] \alpha^{-1} J_{2m+1}(\alpha a) J_{2n+1}(\alpha a). \qquad (8b)$$

It can be shown that $B_{mn} \to 0$ as $d \to 0$, so the matrix formed by A_{mn} is dominated by diagonal elements. To obtain a small d/a approximation to B_{mn}, we rewrite $G(\alpha)$ as

$$G(\alpha) = \frac{\epsilon_r [1 - e^{-2\alpha d}]}{(1 + \epsilon_r) \alpha d} \sum_{n=0}^{\infty} \left(\frac{1 - \epsilon_r}{1 + \epsilon_r} \right)^n e^{-2n\alpha d}. \qquad (9)$$

As such, B_{00} can be evaluated exactly as a summation of elliptic integrals [3] from which we can find its small d approximation. However for general B_{mn}, we resort to other means to find its small d approximation.

First we notice that

$$J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) \sim \frac{2}{\pi \alpha a} [\text{sinusoidal term}] + \frac{(-1)^{n-m}}{\pi \alpha a}$$
$$+ O(\alpha^{-3}), \qquad \alpha \to \infty \qquad (10)$$

and

$$G(\alpha) - 1 \sim -\frac{\alpha d}{\epsilon_r} + O(\alpha^2 d^2), \qquad \alpha d \to 0. \qquad (11)$$

Thus we rewrite (8b) as

$$B_{mn} = C_{mn} + \frac{(-1)^{n-m}}{\pi} \int_0^{\infty} d\alpha \frac{a}{(\alpha a)^2 + A^2} [G(\alpha) - 1] \qquad (12)$$

where

$$C_{mn} = \int_0^{\infty} d\alpha [G(\alpha) - 1] \alpha^{-1} \left\{ J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) \right.$$
$$\left. - \frac{(-1)^{n-m}}{\pi} \frac{\alpha a}{(\alpha a)^2 + A^2} \right\}$$

and A is an arbitrary complex constant. We note that the part in the curly bracket of C_{mn} consists of a sinusoidal term plus a term of $O(\alpha^{-3})$ when $\alpha \to \infty$. Thus the contribution to the integral comes from around $\alpha \approx 0$. In view of (11), we can write C_{mn} as

$$C_{mn} = -\int_0^{\infty} d\alpha \frac{d}{\epsilon_r} \left\{ J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) \right.$$
$$\left. - \frac{(-1)^{n-m}}{\pi} \frac{\alpha a}{(\alpha a)^2 + A^2} \right\}$$
$$+ \int_0^{\infty} d\alpha \left[G(\alpha) - 1 + \frac{\alpha d}{\epsilon_r} \right] \alpha^{-1}$$
$$\times \left\{ J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) - \frac{(-1)^{n-m}}{\pi} \frac{\alpha a}{(\alpha a)^2 + A^2} \right\}. \qquad (13)$$

Both of the above integrals converge. In the limit $d/a \to 0$, the first integral is of $O(d/a)$ while the second integral is of $o(d/a)$ indicating an order higher than the order of d/a and, therefore, less significant. Therefore,

$$C_{mn} \sim -\frac{d}{\epsilon_r} \int_0^{\infty} d\alpha \left\{ J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) \right.$$
$$\left. - \frac{(-1)^{n-m}}{\pi} \frac{\alpha a}{(\alpha a)^2 + A^2} \right\}$$
$$+ o\left(\frac{d}{a} \right), \qquad \frac{d}{a} \to 0,$$
$$= -\frac{d}{\epsilon_r} \lim_{\lambda \to 0} \left\{ \int_0^{\infty} d\alpha (\alpha a)^{-\lambda} J_{2m+1}(\alpha a) J_{2n+1}(\alpha a) d\alpha \right.$$
$$\left. - \frac{(-1)^{n-m}}{\pi} \int_0^{\infty} d\alpha \frac{(\alpha a)^{1-\lambda}}{(\alpha a)^2 + A^2} \right\}. \qquad (14)$$

It can be shown [12, p. 692] that

$$\int_0^\infty d\alpha(\alpha a)^{-\lambda}J_{2m+1}(\alpha a)J_{2n+1}(\alpha a) \sim \frac{(-1)^{n-m}}{\pi\lambda a}\left\{1+\lambda\left[\psi(1)\right.\right.$$

$$-\psi\left(m+n+\frac{3}{2}\right)-\ln 2-\frac{\psi(n-m+1/2)}{2}$$

$$\left.\left.-\frac{\psi(m-n+1/2)}{2}\right]\right\}$$

$$+O(\lambda), \qquad \lambda\to 0 \qquad (15a)$$

where $\psi(x)$ is the psi (digamma) function. Using the recurrence relation of $\psi(x)$, the above approximation can be reduced to

$$\int_0^\infty d\alpha(\alpha a)^{-\lambda}J_{2m+1}(\alpha a)J_{2n+1}(\alpha a) \sim \frac{(-1)^{n-m}}{\pi\lambda a}$$

$$\cdot\left\{1+\lambda\left[\psi(1)-2-\ln 2-2\psi\left(\frac{1}{2}\right)\right.\right.$$

$$\left.\left.-\sum_{\kappa=1}^{m+n}\frac{1}{\kappa+1/2}-\sum_{\kappa=1}^{|m-n|}\frac{1}{\kappa-1/2}\right]\right\}+O(\lambda), \qquad \lambda\to 0. \quad (15b)$$

Also, it can be shown that

$$\frac{(-1)^{n-m}}{\pi}\int_0^\infty d\alpha\frac{(\alpha a)^{1-\lambda}}{(\alpha a)^2+A^2} \sim \frac{(-1)^{n-m}}{\pi\lambda a}$$

$$\cdot\left[1-\lambda\ln A\right]+O(\lambda), \qquad \lambda\to 0. \quad (15c)$$

Therefore, we conclude that

$$C_{mn} \sim -\frac{d}{a\epsilon_r}\frac{(-1)^{n-m}}{\pi}\left[\gamma-2+3\ln 2+\ln A\right.$$

$$\left.-\sum_{\kappa=1}^{m+n}\frac{1}{\kappa+1/2}-\sum_{\kappa=1}^{|m-n|}\frac{1}{\kappa-1/2}\right], \qquad d\to 0 \quad (16)$$

where γ is the Euler's constant. Next, we shall evaluate the second integral of B_{mn}. By making use of (9), it can be rewritten as

$$\frac{(-1)^{n-m}}{\pi}\int_0^\infty d\alpha\frac{a}{(\alpha a)^2+A^2}\left[G(\alpha)-1\right] = \frac{(-1)^{n-m}}{\pi}\frac{a\epsilon_r}{1+\epsilon_r}$$

$$\cdot\sum_{n=0}^\infty\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n\int_0^\infty d\alpha\frac{1}{\alpha d((\alpha a)^2+A^2)}$$

$$\cdot\left[e^{-2n\alpha d}-e^{-2(n+1)\alpha d}\right]-\frac{(-1)^{n-m}}{2A} \quad (17)$$

$$\sim(-1)^{n-m}\frac{d}{a}\frac{4\epsilon_r^2}{1-\epsilon_r^2}\sum_{n=2}^\infty n^2(\ln n)\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n$$

$$+\frac{(-1)^{n-m}}{\pi}\frac{d}{a\epsilon_r}\left[\gamma+\ln\frac{2Ad}{a}-\frac{3}{2}\right]+O(d^2), \qquad \frac{d}{a}\to 0. \quad (18)$$

The small d/a approximation in the above is evaluated in

Appendix A. Hence from (12) we find

$$B_{mn} \sim -\frac{(-1)^{n-m}}{\pi}\frac{d}{a\epsilon_r}\left\{\ln\frac{a}{d}+\ln 4-\frac{1}{2}\right.$$

$$-\frac{4\epsilon_r^3}{1-\epsilon_r^2}\sum_{n=2}^\infty n^2(\ln n)\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n$$

$$\left.-\sum_{\kappa=1}^{m+n}\frac{1}{\kappa+1/2}-\sum_{\kappa=1}^{|m-n|}\frac{1}{\kappa-1/2}\right\}, \qquad \text{as } \frac{d}{a}\to 0. \quad (19)$$

From the above, we deduce that an element of the matrix is

$$A_{mn} \sim \frac{\delta_{mn}}{(2+4n)}-\frac{(-1)^{n-m}}{\pi\epsilon_r}\frac{d}{a}$$

$$\cdot\left\{\ln\frac{a}{d}+\ln 4-\frac{1}{2}-\frac{4\epsilon_r^3}{1-\epsilon_r^2}\sum_{n=2}^\infty n^2\ln n\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n\right.$$

$$\left.-\sum_{\kappa=1}^{m+n}\frac{1}{\kappa+1/2}-\sum_{\kappa=1}^{|m-n|}\frac{1}{\kappa-1/2}\right\}$$

$$+o\left(\frac{d}{a}\right), \qquad \text{as } \frac{d}{a}\to 0. \quad (20)$$

If, instead of choosing an infinite set of basis functions in (5), we choose $N+1$ basis function, then the matrix equation to be solved is

$$\sum_{m=0}^N a_m A_{mn} = E(n), \qquad n=0,1,2,\cdots,N. \quad (21)$$

We define a normalized capacitance $\tilde{C}=Cd/(\pi a^2\epsilon_r\epsilon_0)$. It can be shown, by inverting the matrix in (21) and as a consequence of (6) and (7) that the approximate normalized capacitance is

$$\tilde{C}_N = \frac{\tilde{A}_{oo}}{2\det A|} \quad (22)$$

where $\tilde{A}_{oo}$ is the cofactor of the element A_{oo} of A and A is an $(N+1)\times(N+1)$ matrix. It was shown by Polya and Szegö, or it is apparent as a consequence of variational properties of Galerkin's method [13], that

$$\tilde{C}_0 \leqslant \tilde{C}_1 \leqslant \tilde{C}_2 \leqslant \cdots \leqslant \tilde{C}_n \leqslant \tilde{C}. \quad (23)$$

Since the infinite set of Legendre's polynomial used in (5) is complete, we expect that

$$\tilde{C}= \lim_{N\to\infty}\tilde{C}_N= \lim_{N\to\infty}\frac{\tilde{A}_{oo}}{2\det|A|}. \quad (24)$$

Also we can rewrite

$$\tilde{C}_N = \frac{1}{2}\frac{\tilde{A}_{oo}}{A_{oo}\tilde{A}_{oo}+\sum_{i=1}^N A_{oi}\tilde{A}_{oi}}. \quad (25)$$

From (20), it is seen that $A_{mn}\sim O(1)$ if $n=m$ and $A_{mn}\sim O(d/a)$ if $n\neq m$ when $d/a\to 0$. Thus we conclude that

$A_{oi} \sim O(d/a)$ for $i \neq 0$, and $\Sigma_{i=1}^{N} A_{oi} \tilde{A}_{oi} \sim O(d^2/a^2)$ when $d/a \to 0$. As a consequence

$$\tilde{C}_N \sim \frac{1}{2A_{oo}} + O\left(\frac{d^2}{a^2}\right), \quad \text{as } \frac{d}{a} \to 0. \quad (26)$$

From (20), we obtain

$$C_N \sim 1 + \frac{2d}{\pi\epsilon_r a}\left\{\ln\frac{a}{2d} + \ln 8 - \frac{1}{2} - \frac{4\epsilon_r^3}{1-\epsilon_r^2}\right.$$

$$\left. \cdot \sum_{n=2}^{\infty} n^2 \ln n\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n\right\}$$

$$+ o\left(\frac{d}{a}\right), \quad \text{as } \frac{d}{a} \to 0. \quad (27)$$

We see that for N finite, $\tilde{C}_N$ is asymptotic to the same leading term. From (23), we may be led to the wrong conclusion that $\tilde{C}_\infty = \tilde{C}$ is also asymptotic to the same expression. We have to be cautioned, due to our argument in (10) which is only good for m and n finite, that we cannot extend the validity of (27) to that when $N \to \infty$. Thus (27) is not the correct asymptotic value to $\tilde{C}$ but it is an asymptotic lower bound to $\tilde{C}$. When $\epsilon_r = 1$, we reproduce the asymptotic lower bound obtained by Polya and Szegö [3].

Kirchhoff [1], Hutson [4] and Leppington [5] have all obtained the correct asymptotic value of the capacitance when $\epsilon_r = 1$ with the aid of conformal mapping. When $\epsilon_r > 1$, conformal mapping is not applicable. Thus we shall obtain an approximate formula for the capacitance through a seminumerical approach.

III. NUMERICAL METHOD

The numerical evaluation of the capacitance for the coaxial disks in free space was first carried out in 1941 by Nomura [14]. His results were later checked and corrected where errors existed in 1958 by Cooke [15]. Cooke makes use of the Love–Cooke [16] integral equation which is singular. Nomura expanded the potential in a series of integrals involving Bessel functions. It was found by them that for $d/a < 0.2$, the computation for the capacitance was very laborious and was not carried out.

Itoh and Mittra [7] used Galerkin's method and use of one basis function, found the numerical capacitance of the disk. When $d/a > 0.5$, it was found that the use of Maxwell's function afforded 3 digit accuracy when compared with the Nomura–Cooke results. However, for $d/a < 0.5$, the results deteriorate. Borkar and Yang [9] used Tranter's method to the solution of the dual integral equation but the results obtained are inaccurate for $d/a < 1$ probably due to the difficulties encountered in evaluating the integral (6b). Also the choice of an infinite set of Legendre's polynomial is unsuitable for numerical computation. This is because the singularity of charge distribution at the edge can only be approximated by infinitely many Legendre's polynomials. If we choose $\kappa = 1/2$ in (4),

the corresponding charge distribution from (5) is given by $\sigma(\rho) =$

$$\left\{\begin{array}{l} \sqrt{\frac{2}{a^3}}\left(1 - \frac{\rho^2}{a^2}\right)^{-1/2} \sum_{m=0}^{\infty} a_m \frac{\Gamma(m+1)}{\Gamma(m+1/2)} P_m^{(0,-1/2)}\left(1 - \frac{2\rho^2}{a^2}\right), \\ \hspace{5cm} 0 < \rho < a \quad (28a) \\ \\ 0, \quad \rho > a. \hspace{4cm} (28b) \end{array}\right.$$

We see that such a representation takes into account the charge singularity at the edge which is different from the approach of Borkar and Yang [19] for $d/a < 1$. Notice also that Itoh and Mittra [7] considered only the first term in the series. Analogous to (6b), and noting that $J_{2m+1/2}(\alpha a) = \sqrt{2\alpha a/\pi}\, j_{2m}(\alpha a)$, we define

$$K_{mn} = \int_0^{\infty} d\alpha\, G(\alpha)\alpha j_{2m}(\alpha a) j_{2n}(\alpha a) \quad (29)$$

where $j_p(\alpha)$ is a spherical Bessel function. The integrand is oscillatory and decays algebraically, which makes its evaluation difficult. However, noticing that as $\alpha \to \infty$

$$G(\alpha) \sim \frac{\epsilon_r}{\alpha d[1+\epsilon_r]} + (\text{exponentially small terms}) \quad (30)$$

we rewrite

$$K_{mn} = \int_0^{\infty} d\alpha\left[G(\alpha) - \frac{\epsilon_r}{\alpha d[1+\epsilon_r]}\right]\alpha j_{2m}(\alpha a) j_{2n}(\alpha a)$$

$$+ \frac{\epsilon_r}{d(1+\epsilon_r)}\int_0^{\infty} d\alpha\, j_{2m}(\alpha a) j_{2n}(\alpha a). \quad (31a)$$

The first integrand decays exponentially fast and can be computed efficiently. The second integral can be evaluated explicitly giving

$$\int_0^{\infty} d\alpha\, j_{2m}(\alpha) j_{2n}(\alpha) = \frac{\pi}{2}\frac{\delta_{mn}}{(4n+1)}. \quad (31b)$$

The normalized capacitance using $(N+1)$ basis function is $\tilde{C}_N = 2\tilde{K}_{oo}/\det|K|$.

For numerical computation, it is found that the use of the first two basis function gives 4 digit accuracy when $d/a < 0.4$ and 5 digit accuracy when $d/a > 0.4$ when compared with the Nomura–Cooke results for $\epsilon_r = 1$.

IV. SEMINUMERICAL APPROXIMATION

From the form of the asymptotic lower bound derived by the authors, and the various work of the previous authors, we conclude that an asymptotic approximation to the capacitance when $d/a \to 0$ is of the form

$$C \simeq \frac{a^2 \pi \epsilon_r \epsilon_o}{d}\left\{1 + \frac{2d}{\pi\epsilon_r a}\left[\ln\left(\frac{a}{2d}\right) + g\left(\epsilon_r, \frac{d}{a}\right)\right]\right\} + o(1). \quad (32)$$

An analytic expression for the function $g(\epsilon_r, d/a)$ is difficult to derive. We also note that the asymptotic lower

$$F(\epsilon_r) = -\frac{4\epsilon_r^3}{1-\epsilon_r^2} \sum_{n=2}^{\infty} \left(n^2 \ln n\right)\left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n$$
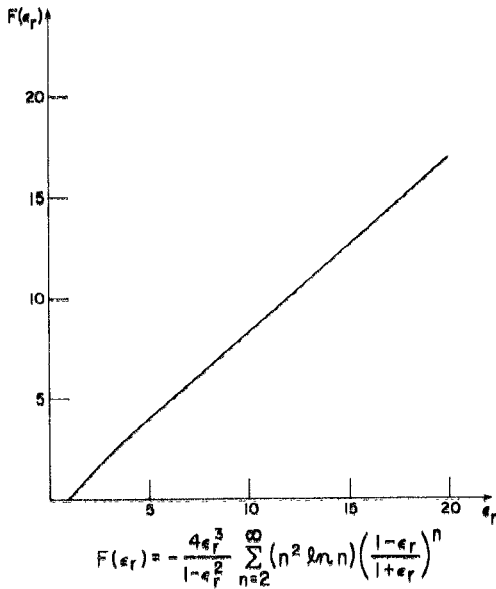
Fig. 2. The function $F(\epsilon_r)$.

bound of the dielectric case differs from the free space case by a term

$$-\frac{4\epsilon_r^3}{1-\epsilon_r^2} \sum_{n=2}^{\infty} n^2 \ln n \left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n$$

which can be shown to be almost a linear function in ϵ_r (Fig. 2). This leads us to believe that the function $g(\epsilon_r, d/a)$ in (32) is also a linear function of ϵ_r. We can fit the numerical curve very well if we also make it a linear function of d/a. Thus the numerical value of the capacitance is computed for $\epsilon_r = 1$ and 8.5 at $d/a = 0.1$ and $d/a = 0.5$. By interpolating between $\epsilon_r = (1, 8.5)$ and $d/a = (0.1, 0.5)$, we obtained the following formula:

$$C \approx \frac{a^2 \pi \epsilon_r \epsilon_o}{d} \left\{ 1 + \frac{2d}{\pi \epsilon_r a} \left[\ln\left(\frac{a}{2d}\right) + (1.41\epsilon_r + 1.77) \right. \right.$$
$$\left. \left. + \frac{d}{a}(0.268\epsilon_r + 1.65) \right] \right\}. \quad (33)$$

We note that the first term is equal to the capacitance by ignoring fringing fields. For the terms accounting for the fringing field effects, the logarithmic term is due to the field emanating from charges at the top surface of the disk and is therefore independent of ϵ_r. The remaining higher order terms are mainly the results of fringing field effects at the edge of the disk which are seen to depend on both ϵ_r and d/a.

V. CONCLUSIONS

We have obtained an approximate formula for the capacitance of a circular microstrip disk in (33) by a seminumerical approach. The asymptotic lower bound has been derived in (27) which reduces to that obtained by Polya and Szegö in the case of $\epsilon_r = 1$. In addition a numerical method is used to compute the highly accurate values for C in order to compare with the various results.
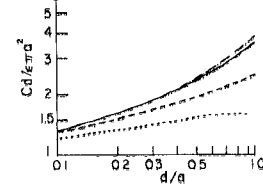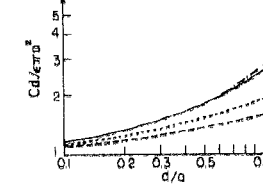

d/a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
NM	1.317	1.580	1.830	2.0751	2.3183	2.5607	2.8034	3.0464	3.2901	3.5346
SNA	1.32	1.57	1.82	2.06	2.32	2.59	2.88	3.16	3.48	3.81
ALB	1.20	1.32	1.40	1.48	1.50	1.53	1.55	1.57	1.57	1.56
KA	1.29	1.49	1.65	1.80	1.93	2.04	2.15	2.25	2.33	2.42

Fig. 3. Normalized capacitance as a function of d/a for $\epsilon_r = 1$.

d/a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
NM	1.180	1.341	1.497	1.6533	1.8100	1.9678	2.1269	2.2873	2.4491	2.6122
SNA	1.18	1.33	1.48	1.64	1.80	1.97	2.15	2.33	2.52	2.72
ALB	1.14	1.25	1.35	1.43	1.51	1.59	1.66	1.73	1.80	1.86

Fig. 4. Normalized capacitance as a function of d/a for $\epsilon_r = 2.65$.
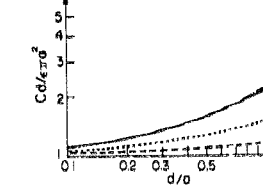
d/a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
NM	1.114	1.228	1.342	1.4579	1.5757	1.6957	1.8177	1.9415	2.0671	2.1942
SNA	1.11	1.23	1.34	1.46	1.58	1.70	1.83	1.96	2.10	2.25
ALB	1.06	1.12	1.17	1.22	1.27	1.31	1.36	1.40	1.45	1.49

Fig. 5. Normalized capacitance as a function of d/a for $\epsilon_r = 9.6$.

In Figs. 3–5 we plot $Cd/\epsilon_r \epsilon_o \pi a^2$ as a function of d/a for the dielectric constants $\epsilon_r = 1$, 2.65, and 9.6. For $\epsilon_r = 1$, we compare our results with Kirchhoff's formula

$$C = \frac{\epsilon_o \pi a^2}{d} \left\{ 1 + \frac{2d}{\pi a} \left[\ln\left(\frac{a}{2d}\right) + \ln(16\pi) - 1 \right] \right\}. \quad (34)$$

When $\epsilon_r \neq 1$, we compare with the formula devised by Shen et al. [8]

$$C = \epsilon_r \frac{\epsilon_o \pi a^2}{d} \left\{ 1 + \frac{2d}{\pi \epsilon_r a} \left[\ln\left(\frac{\pi a}{2d}\right) + 1.7726 \right] \right\}. \quad (35)$$

We note that the seminumerical approximation agrees very well with the numerical result. The error is approximately 1 percent for $d/a \leqslant 0.5$. We also note that the approximations due to Kirchhoff is inaccurate unless $d/a < 0.1$ and due to Shen et al. unless $d/a \ll 0.1$. Shen's approximation incurs an error of about 6 percent when $d/a = 0.1$ and $\epsilon_r = 2.65$.

In general we see that the fringing field effects on the capacitance are not small even for small d/a. For example, when $d/a = 0.1$ and $\epsilon_r = 2.65$, the fringing field gives an 18 percent increase in capacitance. When ϵ_r is large, the fringing field effects become smaller because of the increased trapping of electric flux in the dielectric substrate.

APPENDIX A

We want to evaluate a typical integral in the series given by (17)

$$I = \int_0^\infty \frac{1}{d((\alpha a)^2 + A^2)} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] d\alpha. \quad (A1)$$

First by partial fractioning the rational part of the integrand, the above integral can be written as

$$I = \frac{-1}{A^2 d} \int_0^\infty \frac{1}{2\left(\alpha + \frac{iA}{a}\right)} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] d\alpha$$

$$- \frac{1}{A^2 d} \int_0^\infty \frac{1}{2\left(\alpha - \frac{iA}{a}\right)} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] d\alpha$$

$$+ \frac{1}{A^2 d} \int_0^\infty \frac{1}{\alpha} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] d\alpha. \quad (A2)$$

The first integral can be evaluated exactly in terms of exponential integrals

$$\int_0^\infty \frac{1}{\left(\alpha + \frac{iA}{a}\right)} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] d\alpha = e^{2iAnd/a}$$

$$\cdot E_1(2iAnd/a) - e^{2iA(n+1)d/a} E_1(2kA(n+1)d/a). \quad (A3)$$

The second integral is similar to the first. The third integral can be obtained from (A3) by taking the limiting value of $A \to 0$. As such:

$$\int_0^\infty \frac{1}{\alpha} \left[e^{-2n\alpha d} - e^{-2(n+1)\alpha d} \right] dx = -\ln\left(\frac{n}{n+1}\right). \quad (A4)$$

Therefore

$$I = -\frac{1}{2A^2 d} \left[e^{2iAnd/a} E_1(2iAnd/a) + e^{-2iAnd/a} \right.$$

$$\cdot E_1(-2iAnd/a) - e^{2iA(n+1)d/a} E_1(2iA(n+1)d/a)$$

$$- e^{-2iA(n+1)d/a} E_1(2iA(n+1)d/a)$$

$$\left. + 2\ln\left(\frac{n}{n+1}\right) \right]. \quad (A5)$$

Using the fact that

$$e^{2ixd} E_1(2ixd) + e^{-2ixd} E_1(-2ixd)$$

$$= -2\cos(2xd)\left[\gamma + \ln(2xd) \right]$$

$$- 2\sin(2xd) \sum_{m=0}^\infty \frac{(-1)^m (2xd)^{2m+1}}{(2m+1)!(2m+1)}$$

$$- 2\cos(2xd) \sum_{p=1}^\infty \frac{(-1)^p (2xd)^{2p}}{2p(2p)!} + \pi \sin(2xd)$$

(A6a)

$$\sim -2\left[\gamma + \ln(2x) + \ln d \right](1 - 2x^2 d^2)$$

$$- 6(xd)^2 + 2\pi xd + O(d^3), \qquad d \to 0 \quad (A6b)$$

where γ is the Euler's constant. We can show that

$$I \sim \frac{2d}{a^2} \left[(2n+1)\left(\gamma + \ln\left(\frac{2Ad}{a}\right) - \frac{3}{2} \right) \right.$$

$$\left. - n^2 \ln n + (n+1)^2 \ln(n+1) \right] + \frac{\pi}{Aa} + O(d^2).$$

Then the integral in (17) reduces to

$$\sum_{n=0}^\infty \left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n \int_0^\infty \frac{1}{\alpha d(\alpha^2 + A^2)} \left[e^{-2n\alpha d} - 2^{-2(n+1)\alpha d} \right]$$

$$\cdot d\alpha \sim \frac{4d\epsilon_r}{a^2(1-\epsilon_r)} \sum_{n=2}^\infty n^2 \ln n \left(\frac{1-\epsilon_r}{1+\epsilon_r}\right)^n$$

$$+ \frac{d(1+\epsilon_r)}{a^2 \epsilon_r^2} \left[\gamma + \ln\left(\frac{2Ad}{a}\right) - \frac{3}{2} \right]$$

$$+ \pi \frac{1+\epsilon_r}{2\epsilon_r Aa} + O(d^2), \qquad d \to 0. \quad (A7)$$

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Analysis of a Microstrip Covered with a Lossy Dielectric

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Abstract—The analysis of a microstrip line covered with a low-loss sheet material is presented in this paper. Numerical results show that the characteristics of a microstrip covered with a thick sheet of high dielectric constant are drastically affected. The effect is more pronounced for small values of W/h ratio. A closed-form expression for the dielectric loss of a multilayer structure is derived. The extension of present method to high-loss materials is also discussed.

Numerical and experimental results for effective dielectric constant of a microstrip covered with low- and high-loss sheet materials are compared and found to be in good agreement.

I. INTRODUCTION

THE microstrip line has been widely used as a transmission line [1] as well as a component in microwave-integrated circuits [2]–[4]. Microstrip antennas have found many applications in airborne systems due to their low profile and conformal nature. Recently several papers have appeared in literature describing microwave methods employing microstrip lines for monitoring moisture content in food materials, sheet materials, etc. [5]–[7]. In this case the line was supported on a substrate material of relatively low dielectric constant (< 10) and then covered fully or in part by a "wet" substance of relatively high permittivity (> 15). The fringing field interacts with the substance and produces a change in the attenuation constant of the line. The change in the attenuation constant can be calibrated in terms of moisture content or other parameters which affect the dielectric properties of the material.

When the microstrip line is covered by a sheet material, its characteristics, like characteristic impedance, phase velocity, losses, and Q factor change with the dielectric constant, loss tangent, and thickness of the sheet material. It is interesting and important to study the effect of sheet materials on the characteristics of microstrip lines. Com-

prehensive literature containing analyses of microstrip lines are available [1], [3]. Several methods for the solution of a two-dimensional boundary value problem involving two different media are known, for example, the conformal mapping method [8], the integral equation method [9], [10], the relaxation method [11], and the variational method [12]. The analytical treatment of multiple boundaries is much easier in the variational method than in the conformal mapping or other numerical methods. In the variational method an approximate charge distribution on the strip conductor is assumed and the resulting formulas for capacitance can be expressed in closed form which are convenient for calculation on a digital computer.

In this paper, first the variational method is described for the microstrip covered with a low-loss sheet material then the extension to high-loss materials is discussed. A closed-form expression for the dielectric loss of a multilayer structure is also derived. Numerical results obtained for the characteristic impedance and the phase velocity of a microstrip covered with a lossy sheet can also be used for calculating the resonant frequency of microstrip antennas buried in a lossy medium and to calculate the change in the characteristic impedance and phase velocity values of a microstrip coated with protective layers.

II. THEORY

A. Characteristic Impedance and Phase Velocity

The characteristic impedance Z_0, and the phase velocity v_p, of a TEM transmission line can be written as

$$Z_0 = Z / \sqrt{\epsilon_e} \qquad (1)$$

$$v_p = c / \sqrt{\epsilon_e} \qquad (2)$$

with

$$Z = 1 / C_0 c \qquad (3)$$

$$\epsilon_e = C / C_0 \qquad (4)$$